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LETTER TO THE EDITOR

Boson realisations of Lie algebras and expansion of shift operators

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Abstract. A shift operator expansion is used to extend the range of the boson realisation method of calculation of Lie algebra generator matrix elements to those chains of algebras for which the original technique cannot be applied. The example of $\mathrm{wsp}(4, R) = \mathrm{w}(2) \dotplus \mathrm{sp}(4, R) \supset \mathrm{sp}(4, R)$ is treated in detail and a generalisation to $\mathrm{wsp}(2d, R) = \mathrm{w}(d) \dotplus \mathrm{sp}(2d, R) \supset \mathrm{sp}(2d, R)$ is pointed out.

A few years ago, a procedure was introduced to evaluate the matrix elements of the $\mathrm{sp}(2d, R)$ Lie algebra generators in the $\mathrm{sp}(2d, R) \supset \mathrm{u}(d)$ chain from those of its contracted Lie algebra generators (Rowe *et al* 1984, Deenen and Quesne 1984a). Such a method, based upon boson or coherent-state realisations of Lie algebras (Deenen and Quesne 1982, 1984b, c, 1985, Rowe 1984, Rowe *et al* 1985, Castaños *et al* 1985, 1986), has found many applications to algebras of physical interest (for a list of references see, e.g., Hecht and Suzuki (1987)). It applies to chains of Lie algebras $g \supset h$, for which h contains a Cartan subalgebra of g, and g can be decomposed as the direct sum $g = n_+ \oplus h \oplus n_-$, where the generators in $n_-(n_+)$ correspond to negative (positive) roots and annihilate all the basis states of the subalgebra h irreducible representation (irrep) containing the lowest (highest) weight state of the full algebra g irrep.

However, there exist various physically relevant chains of algebras for which some of the above conditions are not fulfilled. Recently definite progress in the extension of the method to such cases was achieved by Hecht and Suzuki (1987). Their recognition of the important part played by shift operators for $g \supset h$ in such a generalisation raises some interesting problems. In the example studied $\mathrm{usp}(4) \supset \mathrm{usp}(2) \oplus \mathrm{usp}(2)$, the $\mathrm{usp}(2) = \mathrm{su}(2)$ subalgebras indeed make the construction of shift operators quite straightforward, while allowing the Wigner-Racah calculus to be used in their normalisation coefficient calculation by the boson realisation technique. However, for higher-dimensional algebras, the shift operators may be unknown and the Wigner-Racah calculus unavailable. Therefore, techniques similar to the boson realisation technique would be welcome to build shift operators and to evaluate their normalisation coefficient.

The purpose of the present letter is to introduce such methods. Since boson realisations provide a way of reconstructing Lie algebras by expanding their contracted algebras, it is obvious that the converse procedures of shift operator contraction and

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expansion are relevant here. Although shift operator contraction is well known (Wong and Yeh 1979), to the author's knowledge the expansion of the same is new and looks quite promising as an alternative procedure to the known shift operator construction methods (Nagel and Moshinsky 1965, Pang and Hecht 1967, Wong 1967, Bincer 1977, 1978, 1982) which are often rather unwieldy.

The proposed method will be illustrated by calculating the matrix elements of the $wsp(4, R) = w(2) \oplus sp(4, R)$ Lie algebra generators in the physically relevant chain $wsp(4, R) \supset sp(4, R)$, for which the shift operators are unknown. Such a chain is the two-dimensional counterpart of $wsp(6, R) = w(3) \oplus sp(6, R) \supset sp(6, R)$, which finds interesting applications to the description of nuclear collective states (Rowe and Iachello 1983, Quesne 1987a, b).

A basis of $wsp(4, R)$ is provided by the operators $\Lambda_{\alpha\beta} = \Lambda_{\beta\alpha} = (\Lambda_{-\alpha,-\beta})^{\dagger}$, and $V_\alpha = (V_{-\alpha})^{\dagger}$, where Greek indices run over $-2, -1, 1, 2$. Together with the unit operator I, they close under commutation. Their non-zero commutators are given by

$$\begin{aligned} &[\Lambda_{\alpha\beta}, \Lambda_{\gamma\delta}] = g_{\gamma\alpha}\Lambda_{\beta\delta} + g_{\gamma\beta}\Lambda_{\alpha\delta} + g_{\delta\beta}\Lambda_{\gamma\alpha} + g_{\delta\alpha}\Lambda_{\gamma\beta} \\ &[\Lambda_{\alpha\beta}, V_\gamma] = g_{\gamma\alpha}V_\beta + g_{\gamma\beta}V_\alpha \qquad [V_\alpha, V_\beta] = g_{\beta\alpha}I \end{aligned} \tag{1}$$

in terms of the metric tensor $g_{\alpha\beta} = (\alpha/|\alpha|)\delta_{\alpha,-\beta}$. The operators $\Lambda_{\alpha\beta}$ form a basis of $sp(4, R)$, while the operators V_α, together with I, span the Heisenberg-Weyl algebra $w(2)$. In the following, we shall use the notation $D^{\dagger}_{ij} = \Lambda_{ij}$, $D_{ij} = \Lambda_{-i,-j}$, $E_{ij} = \Lambda_{i,-j}$, $B^{\dagger}_i = V_i$, $B_i = V_{-i}$, where Latin indices run over 1 and 2 (Quesne 1987a).

The $wsp(4, R)$ irreps considered here (Quesne 1987a, b) are characterised by two integer or half-odd-integer positive numbers $\Omega_1 \geqslant \Omega_2$, and denoted by $\langle\langle\boldsymbol{\Omega}\rangle\rangle$. By applying the raising generators E_{12}, $D^{\dagger}_{ij}$ and $B^{\dagger}_i$, all basic states of $\langle\langle\boldsymbol{\Omega}\rangle\rangle$ can be obtained from a lowest-weight state (LWS) $|\boldsymbol{\Omega}\rangle$, defined by

$$E_{ii}|\boldsymbol{\Omega}\rangle = \Omega_{3-i}|\boldsymbol{\Omega}\rangle \qquad E_{21}|\boldsymbol{\Omega}\rangle = 0 \qquad D_{ij}|\boldsymbol{\Omega}\rangle = 0 \tag{2}$$

$$B_i|\boldsymbol{\Omega}\rangle = 0. \tag{3}$$

The irrep $\langle\langle\boldsymbol{\Omega}\rangle\rangle$ separates into a direct sum of $sp(4, R)$ irreps $\langle\boldsymbol{\omega}\rangle$ according to the branching rule

$$\langle\langle\boldsymbol{\Omega}\rangle\rangle \downarrow \sum_{\omega_1=\Omega_1}^{\infty} \sum_{\omega_2=\Omega_2}^{\Omega_1} \oplus \langle\boldsymbol{\omega}\rangle \tag{4}$$

where there are no multiplicities. Each $sp(4, R)$ irrep $\langle\boldsymbol{\omega}\rangle$ is a positive discrete series one, whose LWS $|\boldsymbol{\Omega\omega}\rangle$ is defined by equations similar to (2) with Ω_{3-i} replaced by ω_{3-i} and whose remaining states are obtained by applying E_{12} and $D^{\dagger}_{ij}$ on $|\boldsymbol{\Omega\omega}\rangle$.

In the decomposition $wsp(4, R) = n_+ \oplus sp(4, R) \oplus n_-$, the subspace n_- is spanned by the commuting operators B_i, which do not annihilate all the basis states of the $sp(4, R)$ irrep $\langle\boldsymbol{\Omega}\rangle$ containing the $wsp(4, R)$ LWS $|\boldsymbol{\Omega}\rangle$. Then, according to Hecht and Suzuki (1987), we have to consider (i) the raising operators R_i, $i = 1, 2$, for the $wsp(4, R) \supset sp(4, R)$ chain, and (ii) a boson realisation of $wsp(4, R)$ associated with the $wsp(4, R) \supset u(2)$ chain, for which such a problem does not arise.

The raising operators R_i are defined by

$$R_i|\boldsymbol{\Omega\omega}\rangle = N_i(\boldsymbol{\omega})|\boldsymbol{\Omega\omega}^i\rangle \tag{5}$$

where $N_i(\boldsymbol{\omega})$ is some normalisation coefficient, and $\omega^i_j = \omega_j + \delta_{ji}$. Equation (5) is equivalent to the following conditions:

$$[E_{jj}, R_i] = \delta_{3-j,\, i} R_i \tag{6a}$$

$$[E_{21}, R_i] |\boldsymbol{\Omega}\boldsymbol{\omega}\rangle = 0 \tag{6b}$$

$$[D_{jk}, R_i] |\boldsymbol{\Omega}\boldsymbol{\omega}\rangle = 0. \tag{6c}$$

To build R_i in terms of the wsp(4, R) generators, we shall first contract wsp(4, R) $\supset$ sp(4, R) to w(5)$\dotplus$u(2) $\supset$ w(3)$\dotplus$u(2), then build the raising operators $\mathcal{R}_i$ for the latter chain and finally expand $\mathcal{R}_i$ to R_i.

Let us start with the contraction of wsp(4, R). By defining

$$a^\dagger_{ij} = \varepsilon D^\dagger_{ij} \qquad a_{ij} = \varepsilon D_{ij} \qquad \mathscr{E}_{ij} = E_{ij} - (1/2\varepsilon^2)\delta_{ij} I \qquad b^\dagger_i = B^\dagger_i \qquad b_i = B_i \tag{7}$$

in the limit $\varepsilon \to 0$, the commutators (1) become those of the Lie algebra w(5)$\dotplus$u(2), where u(2) is spanned by $\mathscr{E}_{ij}$ and w(5) by the commuting boson creation operators $a^\dagger_{ij} = a^\dagger_{ji}$, $b^\dagger_i$, the corresponding annihilation operators $a_{ij} = a_{ji}$, b_i, and the unit operator. In the same limit, the subalgebra sp(4, R), formally augmented by the addition of the unit operator, contracts to w(3)$\dotplus$u(2), where w(3) is spanned by $a^\dagger_{ij}$, a_{ij} and I. When $\Omega_1 + \Omega_2 = \varepsilon^{-2} \to \infty$, the irrep $\langle\!\langle \boldsymbol{\Omega} \rangle\!\rangle$ of wsp(4, R) contracts to an irrep $\{\{\bar{\boldsymbol{\Omega}}\}\}$ of w(5)$\dotplus$u(2), where $\bar{\Omega}_1 + \bar{\Omega}_2 = 0$ and $\bar{\Omega}_1 - \bar{\Omega}_2 = \Omega_1 - \Omega_2$. Such an irrep has a LWS $|\bar{\boldsymbol{\Omega}}]$, defined by

$$\mathscr{E}_{ii} |\bar{\boldsymbol{\Omega}}] = \bar{\Omega}_{3-i} |\bar{\boldsymbol{\Omega}}] \qquad \mathscr{E}_{21} |\bar{\Omega}] = 0 \qquad a_{ij} |\bar{\boldsymbol{\Omega}}] = 0 \tag{8}$$

$$b_i |\bar{\boldsymbol{\Omega}}] = 0. \tag{9}$$

In its representation space, the u(2) generators ε_{ij} can be written as

$$\boldsymbol{\mathscr{E}} = \boldsymbol{\mathscr{E}}^0 + a^\dagger a + b^\dagger b \tag{10}$$

where the operators ε^0_{ij} span an intrinsic u(2) algebra with a single irrep $[\bar{\boldsymbol{\Omega}}]$. In (10), we use a matrix notion wherein $\boldsymbol{\mathscr{E}}$, $\boldsymbol{\mathscr{E}}^0$, $\boldsymbol{a}^\dagger$, $\boldsymbol{a}$ are 2×2 matrices, $\boldsymbol{b}^\dagger$ is a 2×1 matrix, and $\boldsymbol{b}$ is a 1×2 matrix. The reduction of $\{\{\bar{\boldsymbol{\Omega}}\}\}$ into a sum of w(3)$\dotplus$u(2) irreps $\{\bar{\boldsymbol{\omega}}\}$ is determined by a branching rule similar to equation (4). Each sp(4, R) irrep $\langle\boldsymbol{\omega}\rangle$ contracts to a w(3)$\dotplus$u(2) irrep $\{\bar{\boldsymbol{\omega}}\}$, such that $\bar{\omega}_1 + \bar{\omega}_2 = 0$, and $\bar{\omega}_1 - \bar{\omega}_2 = \omega_1 - \omega_2$. The LWS $|\bar{\boldsymbol{\Omega}}\bar{\boldsymbol{\omega}}]$ of $\{\bar{\boldsymbol{\omega}}\}$ satisfies equations similar to (8) with $\bar{\omega}_{3-i}$ substituted for $\bar{\Omega}_{3-i}$.

Let us then consider the contracted raising operators $\mathcal{R}_i$, i.e. the raising operators for the chain w(5)$\dotplus$u(2) $\supset$ w(3)$\dotplus$u(2), defined by

$$\mathcal{R}_i |\bar{\boldsymbol{\Omega}}\bar{\boldsymbol{\omega}}] = \mathcal{N}_i(\bar{\boldsymbol{\omega}}) |\bar{\boldsymbol{\Omega}}\bar{\boldsymbol{\omega}}^i] \tag{11}$$

where $\mathcal{N}_i(\bar{\boldsymbol{\omega}})$ is a normalisation coefficient different from $N_i(\boldsymbol{\omega})$. They must satisfy conditions similar to equations ($6a$, b, c) with $\mathscr{E}_{jj}$, $\mathscr{E}_{21}$, a_{jk} and $|\bar{\boldsymbol{\Omega}}\bar{\boldsymbol{\omega}}]$ substituted for E_{jj}, E_{21}, D_{jk} and $|\boldsymbol{\Omega}\boldsymbol{\omega}\rangle$, respectively. Such conditions are easily solved yielding

$$\mathcal{R}_1 = b^\dagger_2 \qquad \mathcal{R}_2 = b^\dagger_1(\bar{\mathscr{E}}_{11} - \bar{\mathscr{E}}_{22}) + b^\dagger_2 \bar{\mathscr{E}}_{12} \tag{12}$$

and

$$\mathcal{N}_1(\bar{\boldsymbol{\omega}}) = [(\bar{\omega}_1 - \bar{\Omega}_1 + 1)(\bar{\omega}_1 - \bar{\Omega}_2 + 2)(\bar{\omega}_1 - \bar{\omega}_2 + 2)^{-1}]^{1/2} \tag{13}$$

$$\mathcal{N}_2(\bar{\boldsymbol{\omega}}) = [(\bar{\Omega}_1 - \bar{\omega}_2)(\bar{\omega}_2 - \bar{\Omega}_2 + 1)(\bar{\omega}_1 - \bar{\omega}_2 + 1)]^{1/2}$$

where

$$\bar{\boldsymbol{\mathscr{E}}} = \boldsymbol{\mathscr{E}}^0 + \boldsymbol{b}^\dagger \boldsymbol{b}. \tag{14}$$

With a view to possible extensions of the present method to higher-dimensional algebras, it is worth noting that $\mathcal{R}_i$ and $\mathcal{N}_i(\bar{\boldsymbol{\omega}})$ can also be built by contracting the corresponding operators and coefficients for u(2, 1) $\supset$ u(2) which according to Patera (1973) coincide with those for u(3) $\supset$ u(2) determined by Nagel and Moshinsky (1965).

Let us now turn to the last step in the R_i construction procedure, namely the expansion of $\mathscr{R}_i$ to R_i. Consider the operators

$$R_i = \tilde{R}_i A_i^{(0)} + [G_2, \tilde{R}_i] A_i^{(1)} + [G_4, \tilde{R}_i] A_i^{(2)} \tag{15}$$

where

$$\tilde{R}_1 = B_2^\dagger \qquad \tilde{R}_2 = B_1^\dagger(E_{11} - E_{22}) + B_2^\dagger E_{12} \tag{16}$$

are obtained from equation (12), and

$$G_{2p} = \tfrac{1}{2} \sum_{\alpha_1 \ldots \alpha_{2p}} \sum_{\beta_1 \ldots \beta_{2p}} g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} \ldots g^{\alpha_{2p}\beta_{2p}} \Lambda_{\beta_1\alpha_2} \Lambda_{\beta_2\alpha_3} \ldots \Lambda_{\beta_{2p}\alpha_1} \qquad p = 1, 2 \tag{17}$$

are the sp(4, R) Casimir operators written in terms of the inverse $g^{\alpha\beta}$ of the metric tensor $g_{\alpha\beta}$ and the coefficients $A_i^{(p)}$, $p = 0, 1, 2$, are some functions of E_{11} and E_{22}. These operators (15) obviously satisfy equations (6a) and (6b). The coefficients can now be chosen in such a way that equation (6c) is also fulfilled.

Since $\tilde{R}_i|\boldsymbol{\Omega\omega}\rangle$ is a linear combination of states belonging to the sp(4, R) irreps $\langle\boldsymbol{\omega}^i\rangle$ and $\langle\boldsymbol{\omega}^{-j}\rangle$ where $j = 1, 2$, and $\omega_k^{-j} = \omega_k - \delta_{kj}$, equation (6$c$) is indeed equivalent to the following system of two equations:

$$\alpha_i^{(0)}(\boldsymbol{\omega}) + \sum_{p=1}^{2} [g_{2p}(\boldsymbol{\omega}^{-j}) - g_{2p}(\boldsymbol{\omega})] \alpha_i^{(p)} = 0 \qquad j = 1, 2 \tag{18}$$

for the eigenvalues $\alpha_i^{(p)}(\boldsymbol{\omega})$ of $A_i^{(p)}$ associated with $|\boldsymbol{\Omega\omega}\rangle$. In (18), $g_{2p}(\boldsymbol{\omega})$ denotes the eigenvalue of G_{2p} corresponding to $\langle\boldsymbol{\omega}\rangle$ (Nwachuku 1979). The resolution of (18) leads to the relations

$$\begin{aligned} A_i^{(0)} &= -(2E_{11} - 5)(2E_{22} - 3)(E_{11} + E_{22} - 4) A_i^{(2)} \\ A_i^{(1)} &= -(2E_{11}^2 + 2E_{11}E_{22} + 2E_{22}^2 - 13E_{11} - 11E_{22} + 28) A_i^{(2)} \end{aligned} \tag{19a}$$

where we assume that

$$A_i^{(2)} = -[2(2E_{ii} + 2i - 9)]^{-1} \tag{19b}$$

in order to fix the normalisation of R_i. From the above results we obtain the following explicit expression for R_1:

$$\begin{aligned} R_1 = {} & 2B_2^\dagger(E_{22} - 1)(E_{11} + E_{22} - 3) - [2D_{12}^\dagger(E_{22} - 1) - D_{22}^\dagger E_{12}] B_1 \\ & - D_{22}^\dagger(E_{11} + E_{22} - 2) B_2 \end{aligned} \tag{20}$$

and a similar one for R_2.

The remaining problem to be solved is the evaluation of the normalisation coefficient $N_i(\boldsymbol{\omega})$. For such purpose, let us first consider the following Dyson boson realisation of wsp(4, R):

$$\begin{aligned} &\boldsymbol{E}_D = \boldsymbol{E}^0 + \boldsymbol{a}^\dagger \boldsymbol{a} + \boldsymbol{b}^\dagger \boldsymbol{b} \\ &\boldsymbol{D}_D^\dagger = \boldsymbol{a}^\dagger \qquad \boldsymbol{D}_D = \boldsymbol{a}(\boldsymbol{E}^0 + \boldsymbol{b}^\dagger \boldsymbol{b}) + (\tilde{\boldsymbol{E}}^0 + \tilde{\boldsymbol{b}}\tilde{\boldsymbol{b}}^\dagger)\boldsymbol{a} + \boldsymbol{a}(\boldsymbol{a}^\dagger \boldsymbol{a} - 4\boldsymbol{I}) + \tilde{\boldsymbol{b}}\boldsymbol{b} \\ &\boldsymbol{B}_D^\dagger = \boldsymbol{b}^\dagger \qquad \boldsymbol{B}_D = \tilde{\boldsymbol{b}}^\dagger \boldsymbol{a} + \boldsymbol{b} \end{aligned} \tag{21}$$

associated with the wsp(4, R) $\supset$ u(2) chain (Quesne 1987a, b). Here $\boldsymbol{I}$ denotes the 2×2 unit matrix and $\boldsymbol{E}^0 = \boldsymbol{\mathscr{E}}^0 + \frac{1}{2}(\Omega_1 + \Omega_2)\boldsymbol{I}$. By introducing (21) into equation (20) and its analogue for R_2, we obtain a Dyson realisation $(R_i)_{\mathrm{D}}$ of R_i.

The next step in the $N_i(\boldsymbol{\omega})$ calculation procedure is to convert $(R_i)_{\rm D}$ into a Holstein–Primakoff (HP) realisation

$$(R_i)_{\rm HP} = K^{-1}(R_i)_{\rm D}K \qquad (22)$$

by means of a Hermitian operator K (Deenen and Quesne 1982, 1985, Rowe 1984, Castaños *et al* 1985, Hecht and Suzuki 1987). This amounts to imposing the condition

$$K^{-1}(R_i)_{\rm D}K = K[(R_i^\dagger)_{\rm D}]^\dagger K^{-1}. \qquad (23)$$

Since R_i only acts in the subspace $\mathscr{L}$ of sp(4, R) LWS $|\boldsymbol{\Omega\omega}\rangle$, we may restrict (22) and (23) to the bosonic image of $\mathscr{L}$, namely the space $\mathscr{L}_{\rm B}$ of w(3)$\oplus$u(2) LWS $|\bar{\boldsymbol{\Omega}}\bar{\boldsymbol{\omega}}]$. Then it follows from the branching rule (4) that K is a diagonal operator.

The restriction of $(R_i)_{\rm D}$ and $[(R_i^\dagger)_{\rm D}]^\dagger$ to $\mathscr{L}_{\rm B}$ is carried out by dropping all terms containing an $a_{ij}^\dagger$ or a_{ij} operator on the left or right, respectively, and by using the equivalence $b_2^\dagger b_1 \rightleftarrows -E_{21}^0$. The results can be written as

$$(R_i)_{\rm D} = \mathscr{R}_i F_i^{(1)} \qquad [(R_i^\dagger)_{\rm D}]^\dagger = \mathscr{R}_i F_i^{(2)} \qquad \text{in } \mathscr{L}_{\rm B} \qquad (24)$$

where $\mathscr{R}_i$ are the contracted operators (12), and $F_i^{(1)}$ and $F_i^{(2)}$ are some functions of $E_{jj}^0 + b_j^\dagger b_j, j = 1, 2, \Phi_1^0 = \Sigma_j E_{jj}^0$, and $\Phi_2^0 = \Sigma_{ij} E_{ij}^0 E_{ji}^0$. For $F_1^{(1)}$ and $F_1^{(2)}$, for instance, we obtain

$$\begin{aligned} F_1^{(1)} &= 2(E_{22}^0 + b_2^\dagger b_2 - 1)(E_{11}^0 + E_{22}^0 + b_1^\dagger b_1 + b_2^\dagger b_2 - 3) \\ F_1^{(2)} &= (E_{22}^0 + b_2^\dagger b_2)(E_{22}^0 + b_2^\dagger b_2 + \Phi_1^0 - 5) + \tfrac{1}{2}[\Phi_1^0(\Phi_1^0 - 5) - \Phi_2^0 + 12]. \end{aligned} \qquad (25)$$

By taking the matrix element of equation (23) between the ket $|\bar{\boldsymbol{\Omega}}\bar{\boldsymbol{\omega}}]$ and the bra $[\bar{\boldsymbol{\Omega}}\bar{\boldsymbol{\omega}}^i|$, we obtain the following recursion relations for the eigenvalues $k(\boldsymbol{\Omega}, \boldsymbol{\omega})$ of K:

$$(k(\boldsymbol{\Omega}, \boldsymbol{\omega})/k(\boldsymbol{\Omega}, \boldsymbol{\omega}^i))^2 = f_i^{(2)}(\boldsymbol{\Omega}, \boldsymbol{\omega})/f_i^{(1)}(\boldsymbol{\Omega}, \boldsymbol{\omega}) \qquad i = 1, 2 \qquad (26)$$

where

$$f_i^{(1)}(\boldsymbol{\Omega}, \boldsymbol{\omega}) = 2(\omega_i - i)(\omega_1 + \omega_2 - 3) \qquad f_i^{(2)}(\boldsymbol{\Omega}, \boldsymbol{\omega}) = \textstyle\prod_j (\omega_i + \Omega_j - i - j) \qquad (27)$$

are the eigenvalues of $F_i^{(1)}$ and $F_i^{(2)}$ corresponding to $|\bar{\boldsymbol{\Omega}}\bar{\boldsymbol{\omega}}]$. Equation (26) completely determines the operator K in $\mathscr{L}_{\rm B}$.

By combining equations (22), (24) and (26), an explicit expression for $N_i(\boldsymbol{\omega})$ is finally obtained in terms of the functions (27) and (13). The result is

$$\begin{aligned} N_i(\boldsymbol{\omega}) &= (k(\boldsymbol{\Omega}, \boldsymbol{\omega})/k(\boldsymbol{\Omega}, \boldsymbol{\omega}^i))f_i^{(1)}(\boldsymbol{\Omega}, \boldsymbol{\omega})\mathscr{N}_i(\boldsymbol{\omega}) \\ &= (f_i^{(1)}(\boldsymbol{\Omega}, \boldsymbol{\omega})f_i^{(2)}(\boldsymbol{\Omega}, \boldsymbol{\omega}))^{1/2}\mathscr{N}_i(\boldsymbol{\omega}). \end{aligned} \qquad (28)$$

The knowledge of $N_i(\boldsymbol{\omega})$ or, equivalently, that of K now enables us to calculate the matrix elements of the wsp(4, R) generators $B_i^\dagger$ in $\mathscr{L}$. For such purpose, we may either use the relation

$$\langle\boldsymbol{\Omega\omega}^i|B_{3-i}^\dagger|\boldsymbol{\Omega\omega}\rangle = (k(\boldsymbol{\Omega}, \boldsymbol{\omega})/k(\boldsymbol{\Omega}, \boldsymbol{\omega}^i))[\boldsymbol{\Omega\omega}^i|b_{3-i}^\dagger|\boldsymbol{\Omega\omega}] \qquad (29)$$

or first determine $\langle\boldsymbol{\Omega\omega}^i|\tilde{R}_i|\boldsymbol{\Omega\omega}\rangle$ from (15), (19) and (28) and then deduce $\langle\boldsymbol{\Omega\omega}^i|B_{3-i}^\dagger|\boldsymbol{\Omega\omega}\rangle$ from (16). Both procedures lead to the following results:

$$\begin{aligned} \langle\boldsymbol{\Omega\omega}^1|B_2^\dagger|\boldsymbol{\Omega\omega}\rangle &= [2(\omega_1 - 1)(\omega_1 + \omega_2 - 3)(\omega_1 - \omega_2 + 2)]^{-1/2} \\ &\quad\times[(\omega_1 + \Omega_1 - 2)(\omega_1 + \Omega_2 - 3)(\omega_1 - \Omega_1 + 1)(\omega_1 - \Omega_2 + 2)]^{1/2} \\ \langle\boldsymbol{\Omega\omega}^2|B_1^\dagger|\boldsymbol{\Omega\omega}\rangle &= -[2(\omega_2 - 2)(\omega_1 + \omega_2 - 3)(\omega_1 - \omega_2 + 1)]^{-1/2} \\ &\quad\times[(\omega_2 + \Omega_1 - 3)(\omega_2 + \Omega_2 - 4)(\Omega_1 - \omega_2)(\omega_2 - \Omega_2 + 1)]^{1/2}. \end{aligned} \qquad (30)$$

It is then straightforward to obtain the matrix elements of $B_i^\dagger$ and B_i between any basis states of sp(4, R) irreps (Quesne 1987b). Since the remaining wsp(4, R) generators belong to sp(4, R), their matrix elements are well known (Rowe *et al* 1984, Castaños *et al* 1985, Deenen and Quesne 1985).

In the present letter we have shown that the converse procedures of shift operator contraction and expansion provide a useful tool for implementing the boson realisation technique in a generator matrix element calculation when the original method cannot be applied in a straightforward way. Although written for the wsp(4, R) $\supset$ sp(4, R) chain, the key equations (15) and (24) could be extended to wsp($2d$, R) $\supset$ sp($2d$, R) by including the Casimir operators, $G_6, G_8, \ldots, G_{2d}$, and by considering functions $F_i^{(1)}$ and $F_i^{(2)}$ of $E_{jj}^0 + b_j^\dagger b_j$ and $\Phi_j^0 = \Sigma_{k_1k_2\ldots k_j} E_{k_1k_2}^0 E_{k_2k_3}^0 \ldots E_{k_jk_1}^0$ where $j = 1, \ldots, d$.

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